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A test of hyperscaling for the spin-$\frac{1}{2}$ Ising model in four dimensions

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Abstract. The validity of the hyperscaling relation $2\Delta\text{-}d\nu\text{-}\gamma=0$ is studied for the four-dimensional spin-$\frac{1}{2}$ Ising model. High-temperature series expansions are derived for the fourth-field derivative $\chi_0^{(2)}$ of the free energy on the four-dimensional hyper-face-centred cubic (HFCC) and hyper-body-centred cubic (HBCC) lattices to order 9 and 11 respectively. These are analysed, together with other series already available for the susceptibility χ_0 and correlation length ξ for the HFCC, HBCC and the hyper-simple cubic (HSC) lattices. All these series are found to behave consistently with the asymptotic form $t^{-q}|\ln t|^p$, where t is a reduced temperature variable and q is the appropriate mean-field exponent (so that hyperscaling is satisfied automatically). Our best estimates for p are as follows: $p=0{\cdot}30\pm0{\cdot}05$ (HFCC), $0{\cdot}32\pm0{\cdot}05$ (HBCC) for χ_0 (with $q=1=\gamma$) and $p=0{\cdot}33\pm0{\cdot}05$ for ξ^2 (with $q=1=2\nu$). These estimates are in good agreement with the renormalisation group (RG) prediction of $p=\frac{1}{3}$. Results for $\chi_0^{(2)}$ are more slowly convergent, but are still consistent with $p=\frac{1}{3}$ for $q=4=2\Delta+\gamma$.

1. Introduction

The existence of scaling laws relating critical exponents associated with singularities in thermodynamic functions has long been recognised (Fisher 1967, Kadanoff 1966). These scaling laws fall into two main categories, namely weak scaling relations, and strong or hyperscaling relations which depend explicitly on the spatial dimensionality d of the system. We shall, in this paper, be concerned with the hyperscaling relation

$$2\Delta-d\nu-\gamma=0 \tag{1.1}$$

for $d=4$. Here γ and ν characterise the singularities in the high-temperature zero-field susceptibility χ_0 and correlation length ξ, respectively, while Δ is the gap exponent associated with the high-temperature behaviour of higher-field derivatives of the free energy evaluated in zero field.

One of the earliest studies of the four-dimensional Ising model was that of Fisher and Gaunt (1964), who used a high-temperature series expansion for χ_0 to estimate γ for the hyper-simple cubic (HSC) lattice. Following this work, Moore (1970) derived series expansions for χ_0 and successive moments $M^{(t)}$ of the spin–spin correlation function for three four-dimensional lattices, namely the hyper-face-centred cubic (HFCC), the hyper-body-centred cubic (HBCC) and the HSC lattices. He estimated

$$\gamma=1{\cdot}065\pm0{\cdot}003, \qquad \nu=0{\cdot}536\pm0{\cdot}003. \tag{1.2}$$

The estimate for γ, though smaller than that of Fisher and Gaunt (1964), is greater than the mean-field value of 1. The estimate for ν is also significantly different from $\frac{1}{2}$. Such apparent departures of critical exponents from their mean-field values (in $d=4$) had been attributed (Helfand and Langer 1967, and others) to the presence of logarithmic correction terms modifying the dominant power law singularities. Moore (1970) investigated this possibility, but concluded, on the basis of essentially empirical studies, that, while the $d=4$ series data were suggestive of logarithmic factors, it was not possible to distinguish between their effect and any real departure of the exponents from their mean-field values.

More recently, the renormalisation group (RG) approach has been applied extensively to the study of critical phenomena (Wilson and Kogut 1974, Brézin *et al* 1976). This approach implicitly assumes the validity of hyperscaling for $d=2$, 3 and 4, and predicts that, for $d \geq 4$, all critical exponents assume their mean-field values. It also makes detailed predictions about the nature of the logarithmic terms modifying the power law singularities in $d=4$. According to RG calculations, χ_0, $\chi_0^{(2)}$ and ξ^2 possess the following asymptotic forms in $d=4$:

$$\begin{aligned} &\chi_0 \sim A t^{-\gamma} |\ln t|^p && (\gamma = 1), \\ &\chi_0^{(2)} \sim -B t^{-\gamma-2\Delta} |\ln t|^p && (\Delta = \tfrac{3}{2}), \\ &\xi^2 \sim D^2 t^{-2\nu} |\ln t|^p && (\nu = \tfrac{1}{2}), \end{aligned} \qquad (1.3)$$

where $t = 1 - v/v_c$, and $v = \tanh(J/kT)$ is the usual high-temperature variable. (At the critical temperature T_c, $v = v_c$.) The exponent of the logarithmic term is $p = \frac{1}{3}$ for the Ising model. It is easily verified that the hyperscaling relation $2\Delta - d\nu - \gamma = 0$ is satisfied using the mean-field exponents and $d=4$.

In a recent study, Baker (1977) derived a high-temperature series expansion to order v^9 for $\chi_0^{(2)}$ on the HSC lattice. Using this, together with the available χ_0 and ξ series, he obtained

$$2\Delta - d\nu - \gamma = -0{\cdot}302 \pm 0{\cdot}038. \qquad (1.4)$$

This result is in conflict with RG theory, and led Baker to suggest that the RG theory as implemented at present may not apply to the Ising model. On the basis of series extrapolation studies on $\chi_0^{(2)}/\chi_0$ and $\chi_0 \xi^4$, he concluded that the singularity structure of the high-temperature series is more readily accounted for without the inclusion of logarithmic terms.

Gaunt *et al* (1979), using extended series (to order v^{17}) for χ_0 and $\chi_0^{(2)}$ on the HSC lattice, found that the series exhibited behaviour consistent with (1.3), and concluded that the presence of logarithmic terms could by no means be ruled out. For χ_0 they estimated $p = 0{\cdot}33 \pm 0{\cdot}07$ when $\gamma = 1$, in good agreement with the RG prediction. Though the series for $x_0^{(2)}$ was slowly convergent and proved difficult to extrapolate, $\chi_0^{(2)}/\chi_0$, which according to RG theory should be free of logarithmic terms, did, in fact, extrapolate smoothly to give $2\Delta = 2{\cdot}98$, which differs by only $\frac{2}{3}$% from the mean-field value.

In this paper we extend the treatment of Gaunt *et al* (1979) to the other four-dimensional lattices, namely the HFCC and HBCC lattices. The series coefficients of $\chi_0^{(2)}$ have been calculated by using a direct high-temperature star graph expansion for $\partial^3 F/\partial \rho^3$, where F is the free energy and ρ the magnetisation variable. The method is described in an accompanying paper by McKenzie (1980), and yields a series expansion for $\chi_0^{(2)}/\chi_0^4$ from which $\chi_0^{(2)}$ can be readily calculated. The $\chi_0^{(2)}$ series for the HFCC and

HBCC lattices to order v^9 and v^{11}, respectively, are as follows:

$$\begin{aligned}\chi_0^{(2)} = -2(1 &+ 96v + 5592v^2 + 256\,416v^3 + 10\,186\,536v^4 \\ &+ 367\,573\,152v^5 + 12\,377\,865\,576v^6 \\ &+ 395\,650\,359\,648v^7 + 12\,141\,925\,111\,080v^8 \\ &+ 360\,623\,634\,806\,112v^9 + \ldots)\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\chi_0^{(2)} = -2(1 &+ 64v + 2448v^2 + 73\,920v^3 + 1933\,360v^4 \\ &+ 45\,969\,600v^5 + 1020\,194\,928v^6 \\ &+ 21\,498\,942\,528v^7 + 435\,028\,265\,008v^8 \\ &+ 8520\,986\,823\,232v^9 + 162\,492\,994\,362\,384v^{10} \\ &+ 3030\,234\,317\,201\,600v^{11} + \ldots).\end{aligned} \tag{1.6}$$

We analyse these series together with the available χ_0 and $M^{(2)}$ series (Moore 1970, Gaunt *et al* 1979) for singularities of the form given in (1.3), using a method devised by Guttmann (1978). We assume mean-field exponents (so that hyperscaling holds automatically), and determine the exponents p which best fit the series coefficients. We find that the fits are best when p is close to $\frac{1}{3}$, which is the value predicted by RG theory. Deviations from $p = \frac{1}{3}$ and the relatively large uncertainties in our estimates may be due (Baker and Golner 1977) to the presence of slowly decaying additive correction terms with non-universal amplitudes modifying the dominant singular behaviour given in (1.3).

In § 2 we analyse the χ_0 series, while our analyses of $\chi_0^{(2)}$ and ξ^2 are presented in § 3. Our main conclusions are summarised briefly in § 4.

2. Series analysis of χ_0

For both the HFCC and the HBCC lattices, a Padé approximant analysis (Gaunt and Guttman 1974) of the logarithmic derivative of $\chi_0(v)$ shows a pole on the positive real axis at $v = v_c$, corresponding to the dominant algebraic singularity. For the loose-packed HBCC lattice, there is, in addition, a pole on the negative real axis, near $v = -v_c$, which corresponds to the antiferromagnetic singularity. For both lattices, there is also evidence of a pole–zero sequence along the positive real axis beyond $v = v_c$. This suggests a singularity structure more complicated than a simple pole, and is consistent with the presence of logarithmic terms as in (1.3).

Assuming the asymptotic form given in (1.3), we have used a method due to Guttman (1978) to determine the value of p which best fits the data, with fixed $\gamma = 1$. Denoting by a_n the coefficient of v^n in χ_0 for the HFCC lattice, we form the ratios $r_n(=a_n/a_{n-1})$. Defining the mimic function $f(v)$ by

$$v^{-p^*}f(v) \equiv v^{-p^*}(1-v)^{-1}|\ln(1-v)|^{p^*} = \sum_{n\geq 0} b_n v^n, \tag{2.1}$$

we form the ratios $r_n^*(=b_n/b_{n-1})$. The ratios r_n and r_n^* are compared by examining the sequence $R_n(=r_n/r_n^*)$. If the asymptotic form given in (1.3) adequately describes χ_0, the sequence R_n should tend to v_c^{-1} with zero slope as $n \to \infty$, for the correct choice of p^*. Also, the sequence $n(R_n v_c - 1)$ should tend to zero as $n \to \infty$. Higher-order correction terms in $n^{-2}, n^{-3}, \ldots$ can be taken into account by forming linear and quadratic extrapolants to R_n.

For the HBCC lattice, we follow a slightly modified procedure which reduces interference from the antiferromagnetic singularity at $-v_c$. We transform to a new variable x defined by

$$x = \frac{2v}{(1+v/v_c)}. \tag{2.2}$$

This transformation maps the singularity $v = -v_c$ to ∞, but leaves that at v_c unchanged. The initial value of v_c for performing the transformation may be obtained from ratio analysis of χ_0. Having derived the series in the x variable, we follow the procedure outlined above to estimate x_c and p. The true critical point v_c is recovered by using (2.2).

To get some idea of the rate of convergence, we present in table 1 the ratios R_n, together with their linear and quadratic extrapolants, and the sequences $n(R_n x_c - 1)$, together with their linear extrapolants, for the HBCC lattice. We make the estimates

$$\begin{aligned} v_c^{-1} &= 14{\cdot}515 \pm 0{\cdot}01, \qquad p = 0{\cdot}32 \pm 0{\cdot}05 \ (\text{HBCC}), \\ v_c^{-1} &= 21{\cdot}99 \pm 0{\cdot}01, \qquad p = 0{\cdot}30 \pm 0{\cdot}05 \ (\text{HFCC}). \end{aligned} \tag{2.3}$$

For the HSC lattice, Gaunt *et al* (1979) estimate

$$v_c^{-1} = 6{\cdot}7315 \pm 0{\cdot}0015, \qquad p = 0{\cdot}33 \pm 0{\cdot}07 \ (\text{HSC}). \tag{2.4}$$

Table 1. HBCC lattice. Analysis of transformed χ_0 series assuming $v_c^{-1} = 14{\cdot}515$.

p	n	R_n	Linear extrapolant	Quadratic extrapolant	$n(R_n x_c - 1)$	Linear extrapolant
0·27	6	14·4641	14·6118	14·5231	−0·0210	0·0123
	7	14·4813	14·5843	14·5154	−0·0162	0·0124
	8	14·4919	14·5663	14·5124	−0·0127	0·0120
	9	14,4988	14·5541	14·5114	−0·0100	0·0115
	10	14·5035	14·5456	14·5114	−0·0079	0·0110
	11	14·5068	14·5394	14·5116	−0·0062	0·0106
0·32	6	14·4241	14·6209	14·5287	−0·0376	−0·0011
	7	14·4481	14·5920	14·5197	−0·0323	−0·0005
	8	14·4637	14·5729	14·5157	−0·0283	−0·0004
	9	14·4474	14·5598	14·5141	−0·0252	−0·0005
	10	14·4820	14·5506	14·5136	−0·0227	−0·0007
	11	14·4876	14·5439	14·5135	−0·0208	−0·0009
0·37	6	14·3838	14·6297	14·5346	−0·0542	−0·0147
	7	14·4146	14·5995	14·5241	−0·0484	−0·0135
	8	14·4352	14·5794	14·5191	−0·0440	−0·0129
	9	14·4497	14·5655	14·5169	−0·0405	−0·0126
	10	14·4603	14·5556	14·5159	−0·0377	−0·0125
	11	14·4683	14·5483	14·5154	−0·0354	−0·0124

3. Analysis of $\chi_0^{(2)}$ and ξ^2

We use the estimates of v_c given in (2.3) and (2.4) obtained from the χ_0 series, and calculate p for the $\chi_0^{(2)}$ and ξ^2 series using the method outlined in the previous section.

The correlation length ξ is defined by

$$\xi^2 = M^{(2)}/2d\chi_0, \tag{3.1}$$

where $M^{(2)}$ is the second moment of the spin–spin correlation function. We use the $M^{(2)}$ series given by Moore (1970) to obtain ξ^2 to order v^{10} for the HFCC lattice and order v^{11} for the HSC and HBCC lattices. We have also analysed $(1+M^{(2)})/2d\chi_0$, but find no significant improvement in the rate of convergence.

For all three lattices, the ξ^2 series behaves consistently with the asymptotic form in (1.3), and fixing $q = 2\nu = 1$ we make the estimate

$$p = 0{\cdot}33 \pm 0{\cdot}05. \tag{3.2}$$

The sequences R_n and $n(R_n x_c - 1)$ for the HSC lattice, together with their extrapolants, are presented in table 2.

The $\chi_0^{(2)}$ series given in (1.5) and (1.6) have been analysed using the same technique. In table 3 we give our results for the HFCC lattice. The sequences R_n show a great deal of curvature, but Neville table analysis of R_n does yield an estimate of v_c close to that given in (2.3). For p we estimate

$$p = 0{\cdot}32 \pm 0{\cdot}05. \tag{3.3}$$

From (1.3) we see that, although χ_0 and $\chi_0^{(2)}$ both possess logarithmic correction terms, the function $\chi_0^{(2)}/\chi_0$ should be free of logarithms and should exhibit a simple power law singularity with an exponent of 3. We therefore form Padé approximants to

Table 2. HSC lattice. Analysis of transformed ξ^2 series assuming $v_c^{-1} = 6{\cdot}7315$.

p	n	R_n	Linear extrapolant	Quadratic extrapolant	$n(R_n x_c - 1)$	Linear extrapolant
0·28	5	6·7232	6·7987	6·7415	−0·0062	0·0338
	6	6·7320	6·7763	6·7314	0·0005	0·0338
	7	6·7364	6·7625	6·7280	0·0051	0·0327
	8	6·7386	6·7536	6·7270	0·0084	0·0314
	9	6·7396	6·7477	6·7268	0·0108	0·0300
	10	6.7400	6·7435	6·7268	0·0126	0·0286
0·33	5	6·6984	6·8042	6·7456	−0·0246	0·0186
	6	6·7121	6·7809	6·7343	−0·0173	0·0194
	7	6·7199	6·7664	6·7301	−0·0121	0·0190
	8	6·7245	6·7569	6·7286	−0·0083	0·0182
	9	6·7274	6·7505	6·7281	−0·0055	0·0172
	10	6·7293	6·7460	6·7279	−0·0033	0·0161
0.38	5	6·6766	6·8087	6·7493	−0·0408	0·0051
	6	6·6946	6·7848	6·7368	−0·0329	0·0067
	7	6·7053	6·7697	6·7320	−0·0272	0·0068
	8	6·7121	6·7598	6·7301	−0·0230	0·0064
	9	6·7167	6·7530	6·7293	−0·0198	0·0058
	10	6·7198	6·7482	6·7289	−0·0173	0·0050

the logarithmic derivative of $\chi_0^{(2)}/\chi_0$, and these are presented in table 4. The estimates of v_c are consistent with (2.3), for both lattices, and the exponents are close to 3.

The critical amplitudes A, B and D for the HFCC lattice have been estimated by evaluating, at $y=1$, the Padé approximants to the series for

$$y^{1/3}F(y)(1-y)^q|\ln(1-y)|^{-1/3}, \tag{3.4}$$

where $y=v/v_c$, $F(y)$ denotes the χ_0, $\chi_0^{(2)}$ or ξ^2 series in the y variable, and q is the appropriate mean-field exponent. For the loose-packed lattices, we have followed an analogous procedure after first transforming to the x variable defined in (2.2). Our

Table 3. HFCC lattice. Analysis of $\chi_0^{(2)}$ series assuming $v_c^{-1}=21{\cdot}99$.

p	n	R_n	Linear extrapolant	Quadratic extrapolant	$n(R_nv_c-1)$	Linear extrapolant
	4	22·2995	21·8734	21·8027	0·0563	0·0404
	5	22·2118	21·8609	21·8423	0·0504	0·0269
0·28	6	22·1544	21·8679	21·8817	0·0449	0·0171
	7	22·1153	21·8803	21·9115	0·0399	0·0099
	8	22·0876	21·8934	21·9326	0·0355	0·0047
	9	22·0673	21·9054	21·9475	0·0316	0·0009
	4	22·2431	21·8515	21·7932	0·0460	0·0271
	5	22·1637	21·8462	21·8384	0·0395	0·0134
0·32	6	22·1127	21·8577	21·8807	0·0335	0·0034
	7	22·0785	21·8732	21·9119	0·0282	−0·0037
	8	22·0547	21·8883	21·9338	0·0236	−0·0088
	9	22·0377	21·9018	21·9490	0·0195	−0·0125
	4	22·1869	21·8294	21·7835	0·0358	0·0139
	5	22·1158	21·8314	21·8345	0·0286	−0·0002
0·36	6	22·0711	21·8475	21·8795	0·0221	−0·0103
	7	22·0418	21·8660	21·9122	0·0165	−0·0174
	8	22·0220	21·8832	21·9348	0·0116	−0·0224
	9	22·0082	21·8981	21·9505	0·0075	−0·0260

Table 4. D log Padé analysis of $\chi_0^{(2)}/\chi_0$ series for (a) HFCC and (b) HBCC lattices.

(a) HFCC lattice n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
2	21·8706 (−3·044)	21·9589 (−3·002)	22·0083 (−2·972)
3	22·0181 (−2·964)	22·0016 (−2·977)	21·9822 (−3·000)
4	21·9946 (−2·983)	21·9919 (−2·986)	

(b) HBCC lattice n	$[n-1/n]$	$[n/n]$	$[n+1/n]$
2	14·3846 (−3·102)	14·3013 (−3·154)	14·5693 (−2·934)
3	14·3640 (−3·111)	14·5072 (−2·993)	14·5215 (−2·977)
4	14·5286 (−2·968)	14·5193 (−2·980)	14·6115 (−3·713)†
5	14·5164 (−2·984)	14·5133 (−2·990)	

† Defective approximant (Gaunt and Guttmann 1974).

estimates of the critical amplitudes for all three lattices are presented in table 5. These values were calculated using the central v_c in (2.3) and (2.4). Uncertainties in v_c produce further uncertainties in the amplitudes of the same order as the confidence limits quoted in table 5. However, the presence of the additive correction terms referred to at the end of § 1 may mean that our estimates (table 5) are 'effective' amplitudes rather than the dominant ones.

Table 5. Critical amplitudes.

Lattice	A	B	D
HSC	0.91 ± 0.01	2.30 ± 0.07	0.374 ± 0.003
HBCC	0.81 ± 0.02	2.02 ± 0.08	0.238 ± 0.004
HFCC	0.85 ± 0.02	2.18 ± 0.07	0.200 ± 0.002

4. Conclusions

We have investigated high-temperature series expansions for χ_0, $\chi_0^{(2)}$ and ξ^2 for three different lattices for the $d = 4$ Ising model, and found them consistent with the asymptotic forms predicted by RG theory. Specifically, we have assumed mean-field critical exponents, so that the hyperscaling relation $2\Delta - d\nu - \gamma = 0$ is satisfied automatically, and estimated the exponents p pertaining to the confluent logarithmic correction terms. Our estimates for p are in reasonably close agreement with the value of $\frac{1}{3}$ predicted by RG theory.

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